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Heavy quark distribution functions from QCD

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Abstract

The structure function of a heavy quark in a heavy hadron in electroproduction is calculated from QCD. Its deviation from the parton model prediction (a delta function) is shown to be governed by a shape function, similar to the one recently proposed by Neubert and Bigi *et al.* to describe the end-point behaviour of the lepton spectrum in semileptonic $B \rightarrow X_u e \bar{\nu}$ decays and the line shape in $B \rightarrow X_s \gamma$. Two new sum rules are derived which extend the Bjorken-Dunietz-Taron sum rule by including corrections suppressed by the inverse heavy quark mass.

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1. There have been recently significant advances in the theoretical understanding of the inclusive decays of hadrons containing one heavy quark. It has been shown that a number of model-independent predictions can be obtained, for example for the lepton energy spectrum in semileptonic decays of B mesons or the photon energy spectrum in the decay $B \rightarrow X_s \gamma$ [1, 2]. The formal tool which was used is an adaptation of the operator product expansion method (OPE) to deal with this specific situation [3], in which the large mass scale (the heavy quark mass) is transferred from the matrix elements of the operators to the Wilson coefficients. From an intuitive point of view, the results thus obtained can be interpreted as corrections to the naive free-quark decay predictions, due to a “Fermi motion” of the heavy quark inside the hadron. Initially introduced by Neubert to describe the end-point region behaviour of the lepton spectrum in $B \rightarrow X_u e \bar{\nu}$ [4], the concept of a “shape function” turned out to be a process-independent parametrization of the “Fermi motion”, incorporating features which cannot be reproduced by a quantum-mechanical description in terms of a wavefunction [5, 6, 7].

In this paper we discuss the b quark structure functions in electroproduction on a B meson and show that they can be calculated in QCD into an expansion in powers of $\bar{\Lambda}/m_b$. We give explicit results for these function to second order in this parameter. To leading order in this expansion the parton model prediction is obtained, which is a simple delta function. The corrections to it have a singular form consisting of delta functions and its derivatives. It is shown that the most singular contributions can be summed up to all orders in $1/m_b$ into a “shape function” which extends beyond the parton model region and which can be considered a good approximation to the real distribution function.

In Section 4, two generalizations of the Bjorken sum rule are derived [8]. The sum rules result from equating the total contribution to the moments of the heavy quark structure functions in electroproduction on a heavy meson (as obtained by the OPE analysis) to the sum of the individual contributions of the resonances. Both these quantities can be arranged as an expansion in $1/m_Q$, the inverse heavy quark mass, and at the leading order in this expansion the usual Bjorken sum rule is recovered. At higher orders in $1/m_Q$ the resulting sum rules are new and can be used to set constraints on the nonperturbative subleading form-factors.

The spirit of the approach and the nature of the approximations which are made are related to those recently used [1] to calculate the inclusive semileptonic decays of heavy hadrons containing a heavy quark. However, the formalism used here is more similar to the one familiar from deep-inelastic scattering. Instead of modifying the operator-product expansion (OPE) so that the heavy-mass dependence resides from beginning in the coefficient functions, we will prefer to perform the OPE in the full theory (QCD), and to expand the resulting matrix elements in powers of $1/m_Q$. The equivalence of the two approaches is proved on an explicit example to order $1/m_Q^2$.

The heavy quark structure functions in electroproduction on a heavy meson have been considered previously by Jaffe and Randall [9]. In this paper we go further by giving explicit expressions for quantities which were just parametrized in [9]. Also, some of the results described in Section 3 have been obtained by Neubert [5] in a light-cone language. Our derivation is perhaps more direct as it proceeds directly from the usual definitions of the structure functions.

2. Let us consider the forward Compton scattering matrix element on a heavy meson B

$$T_{\mu\nu}(q \cdot p, Q^2) = -i \int d^4x e^{-iq \cdot x} \langle B(v) | T(\bar{b}\gamma_\mu b)(x) (\bar{b}\gamma_\nu b)(0) | B(v) \rangle, \quad (1)$$

where $p = m_B v$ is the B-meson momentum and $Q^2 = -q^2 > 0$ is the photon virtuality. The meson state $|B(v)\rangle$ has the usual normalization $2m_B v_0$. This situation is similar to what one usually has in deep-inelastic scattering, in that the virtual photon is spacelike. However, in distinction to the latter, we will not require that $Q^2 \gg m_B^2$, but will take $Q^2, m_B^2 \gg \Lambda_{QCD}$ (with the exception of Section 3, where we take temporarily $Q^2 \rightarrow \infty$). We will keep therefore all contribution of order m_B^2/Q^2 and organize the result as an expansion in powers of Λ/m_B .

$T_{\mu\nu}$ can be decomposed into covariants defined in the usual way by

$$T_{\mu\nu}(q \cdot p, Q^2) = -T_1(g_{\mu\nu} - \frac{q_\mu q_\nu}{q^2}) + \frac{T_2}{m_B^2}(p_\mu - q_\mu \frac{q \cdot p}{q^2})(p_\nu - q_\nu \frac{q \cdot p}{q^2}) - \frac{1}{2m_B^2} T_3 i\epsilon_{\mu\nu\alpha\beta} p^\alpha q^\beta. \quad (2)$$

In our case, because the electromagnetic interactions are parity-invariant, $T_3 = 0$. The analytic structure of the scalar form-factors $T_{1,2}(q \cdot p, Q^2)$ in the $q \cdot p$ complex plane for fixed Q^2 is shown in Fig.1. It has two poles at $q \cdot p = \pm Q^2/2$, corresponding to the elastic scattering process $\gamma^* B \rightarrow \gamma^* B$ and two cuts for $|q \cdot p| > 1/2[Q^2 + (m_B + m_\pi)^2 - m_B^2]$. $T_{1,2}$ are even functions in the variable $q \cdot p$:

$$T_{1,2}(-q \cdot p, Q^2) = T_{1,2}(q \cdot p, Q^2). \quad (3)$$

The discontinuities of $T_{1,2}$ across the cuts are related to the hadronic tensor $W_{\mu\nu}$ which describes the inclusive process $e^- B \rightarrow e^- X_b$ (in the approximation that only the b-quark is struck):

$$W_{\mu\nu} = \frac{1}{4\pi} \sum_X \langle B | (\bar{b}\gamma_\mu b)(0) | X \rangle \langle X | (\bar{b}\gamma_\nu b)(0) | B \rangle (2\pi)^4 \delta(p_X - p + q) \quad (4)$$

which has a decomposition into covariants similar to $T_{\mu\nu}$. We have

$$\text{disc } T_{\mu\nu}(q, p) = 4\pi i W_{\mu\nu} \quad \text{for } v \cdot q < 0 \quad (5)$$

$$\text{disc } T_{\mu\nu}(q, p) = -\text{disc } T_{\mu\nu}(-q, p) \quad \text{for } v \cdot q > 0. \quad (6)$$

To identify the operators which appear in the OPE (1), the usual method is to take the matrix element of the time-ordered product between free b -quark states. This gives

$$-i \int d^4x e^{-iq \cdot x} \langle b(r) | T(\bar{b} \gamma_\mu b)(x) (\bar{b} \gamma_\nu b)(0) | b(r) \rangle = \quad (7)$$

$$\bar{u}(r) \gamma_\mu \frac{\not{r} - \not{q} + m_b}{(r - q)^2 - m_b^2} \gamma_\nu u(r) + \bar{u}(r) \gamma_\nu \frac{\not{r} + \not{q} + m_b}{(r + q)^2 - m_b^2} \gamma_\mu u(r)$$

with r the momentum of the quark. This has to be expanded in powers of r and the result identified with the matrix elements of a sequence of operators of the type $\bar{b} \cdots b$ with an increasing number of derivatives acting on the fermion field (one for each power of r). At leading order in the coupling g , we can replace r^2 in the denominators of (7) by m_b^2 , since it generates contributions of the form $(iD)^2 b$ which equals (in matrix elements) $m_b^2 b$ in virtue of the equation of motion for the b field. The final result is

$$-i \int d^4x e^{-iq \cdot x} T(\bar{b} \gamma_\mu b)(x) (\bar{b} \gamma_\nu b)(0) = \quad (8)$$

$$-\frac{4}{Q^4} \sum_{n=0}^{\infty} \left(\frac{2}{Q^2} \right)^{2n} \{ Q^2 [g_{\mu\mu_1} g_{\nu\mu_2} - \frac{1}{2} g_{\mu\nu} g_{\mu_1\mu_2}] + q_\mu q_{\mu_1} g_{\nu\mu_2} + q_{\mu_1} q_\nu g_{\mu\mu_2} - q_{\mu_1} q_{\mu_2} g_{\mu\nu} \}$$

$$q_{\mu_3} \cdots q_{\mu_{2n+2}} P^{\mu_1\mu_2 \cdots \mu_{2n+2}} - \frac{2m_b}{Q^2} g_{\mu\nu} \sum_{n=0}^{\infty} \left(\frac{2}{Q^2} \right)^{2n} q_{\mu_1} \cdots q_{\mu_{2n}} R^{\mu_1\mu_2 \cdots \mu_{2n}} .$$

Here P and R are defined to be the operators

$$P_{\mu_1\mu_2 \cdots \mu_n} = \frac{1}{n!} (\bar{b} \gamma_{\mu_1} (iD)_{\mu_2} \cdots (iD)_{\mu_n} b + \text{permutations of the indices}) \quad (9)$$

$$Q_{\mu_1\mu_2 \cdots \mu_n} = \frac{1}{n!} (\bar{b} (iD)_{\mu_1} \cdots (iD)_{\mu_n} b + \text{permutations of the indices}) . \quad (10)$$

In deriving (8) all operators but the completely symmetric ones have been neglected, as their matrix elements in an unpolarized hadronic state vanish. This is the relation on which all our discussion will be based. We will take its expectation value between heavy hadron states and the matrix elements of the operators P and R will be expressed as power series in $1/m_b$ by using heavy quark effective theory methods. But before doing so, we pause for a moment to discuss the method by which the information obtained with the help of the OPE is related to physical quantities.

Previous studies [1] of the inclusive semileptonic decays of hadrons containing one heavy quark have simply identified the hadronic tensor with the discontinuity of the amplitude $T_{\mu\nu}$ calculated using perturbative QCD. The results thus obtained are highly singular and must be understood in the sense of duality, when integrated with an arbitrary smooth function. In this paper we will take a different approach and

will relate the moments of the structure function to the coefficients of $v \cdot q$ appearing into an expansion of $T_{\mu\nu}$ around the origin in the $v \cdot q$ -complex plane with the help of a dispersion relation [10]. In the end the results obtained with the two methods will be seen to be identical. The principle of the method is the following: for very large Q^2 , the two cuts on Fig.1 are well separated and one can use perturbative QCD to reliably compute $T_{\mu\nu}$ around the point $q \cdot p=0$. In this region each of the invariant functions $T_{1,2}$ can be expanded in a power series

$$T_i(q \cdot p, Q^2) = \sum_{n=0}^{\infty} T_{in}(Q^2) \left(\frac{2q \cdot p}{Q^2} \right)^n. \quad (11)$$

The coefficients can be expressed as a contour integral

$$T_{in}(Q^2) = \frac{1}{2\pi i} \int_C dz \frac{T_i(z, Q^2)}{z^{n+1}} \quad (12)$$

where the contour C is a small circle around the origin. It can be deformed to the shape shown in Fig.1, where it picks contributions from the discontinuities across the cuts. The contribution of the large circle at infinity will be neglected. By making use of (5,6), the contour integral in (12) can be written as

$$T_{in}(Q^2) = -2 \int_0^1 dx x^{n-1} W_i(x, Q^2) \quad (13)$$

where $x = Q^2/(2q \cdot p)$ is the Bjorken variable. This is the desired relation which expresses the moments of the hadronic tensor in terms of quantities calculable theoretically, the expansion coefficients T_{in} .

We turn now to the evaluation of the transition matrix element $T_{\mu\nu}$, by taking the matrix element of the OPE expansion (8) between heavy meson states. This leads us to the problem of calculating the matrix elements

$$\langle B(v) | P_{\mu_1 \mu_2 \dots \mu_n} | B(v) \rangle \quad \langle B(v) | R_{\mu_1 \mu_2 \dots \mu_n} | B(v) \rangle. \quad (14)$$

They can be expressed as power series in the inverse heavy quark mass $1/m_b$ by making use of heavy quark effective theory methods. First, the b-quark field is written as

$$b(x) = e^{-im_b v \cdot x} B(x) \quad (15)$$

where

$$B(x) = \left[1 + \frac{i \not{p}_\perp}{2m_b} + \frac{1}{4m_b^2} \left(v \cdot D \not{p}_\perp - \frac{1}{2} \not{p}_\perp^2 \right) + \dots \right] h_v^{(b)}(x). \quad (16)$$

The heavy quark field $h_v^{(b)}$ satisfies $\not{v} h_v^{(b)} = h_v^{(b)}$.

Let us consider for beginning the matrix element of P . For simplicity we exemplify the calculation on the first term in (9), the final result having to be obtained by symmetrization with respect to all indices. First we write

$$\begin{aligned} \langle B(v) | \bar{b} \gamma_{\mu_1} (iD)_{\mu_2} \cdots (iD)_{\mu_n} b | B(v) \rangle = \\ \langle B(v) | \bar{B} \gamma_{\mu_1} (m_b v + iD)_{\mu_2} (m_b v + iD)_{\mu_3} \cdots (m_b v + iD)_{\mu_n} B | B(v) \rangle. \end{aligned} \quad (17)$$

To leading order in the large mass scale this is equal to

$$2m_B m_b^{n-1} v_{\mu_1} v_{\mu_2} \cdots v_{\mu_n}, \quad (18)$$

which is already symmetric in all indices. Analogously, the matrix element of R can be written as

$$\begin{aligned} \langle B(v) | \bar{B} (m_b v + iD)_{\mu_1} (m_b v + iD)_{\mu_2} \cdots (m_b v + iD)_{\mu_n} B | B(v) \rangle = \\ 2m_B m_b^n v_{\mu_1} v_{\mu_2} \cdots v_{\mu_n} + \text{terms suppressed by } 1/m_b. \end{aligned} \quad (19)$$

If we insert these expressions for the matrix elements in the equation (8) for $T_{\mu\nu}$, the following results are obtained for the invariant transition amplitudes

$$T_1(q \cdot p, Q^2) = -2 \sum_{n=0}^{\infty} \left(\frac{m_b}{m_B} \right)^{2n+1} \left(\frac{2m_B v \cdot q}{Q^2} \right)^{2n+2} \quad (20)$$

$$T_2(q \cdot p, Q^2) = -8 \frac{m_B^2}{Q^2} \sum_{n=0}^{\infty} \left(\frac{m_b}{m_B} \right)^{2n+1} \left(\frac{2m_B v \cdot q}{Q^2} \right)^{2n}. \quad (21)$$

If we define the more conventional structure functions $F_1(x, Q^2) = W_1(x, Q^2)$ and $F_2(x, Q^2) = q \cdot p / m_B^2 W_2(x, Q^2)$, the following predictions are obtained for their moments, with the help of Eq.(13)

$$\int_0^1 dx x^{n-1} F_1(x, Q^2) = \left(\frac{m_b}{m_B} \right)^{n-1} = \left(1 - \frac{\bar{\Lambda}}{m_B} \right)^{n-1}, \quad n = 2, 4, 6, \dots \quad (22)$$

$$\int_0^1 dx x^{n-1} F_2(x, Q^2) = 2 \left(\frac{m_b}{m_B} \right)^n = 2 \left(1 - \frac{\bar{\Lambda}}{m_B} \right)^n, \quad n = 1, 3, 5, \dots \quad (23)$$

The evenness property of $T_{1,2}$ (3) prevents us from extracting the moments (22,23) for other values of n than those indicated. However, these can be obtained by analytic continuation, that is, by simply assuming that these relations hold true for all values of n .

Some comments can be made about this result: i) the integral of $F_1(x, Q^2)$ equals unity ($n=1$), which corresponds to the parton interpretation of this function as being the probability for the heavy quark to be found with a fraction x of the

total momentum; ii) as expected, the Callan-Gross relation $F_L = F_2 - 2xF_1 = 0$ is satisfied; iii) the shape of the distribution function is a simple delta-function

$$F_1(x, Q^2) = \delta(x - \frac{m_b}{m_B}) \quad (24)$$

which is the naive parton model result, although modified to take into account the binding energy of the heavy quark in the hadron.

Let us now go further and examine the effect of retaining more terms in the matrix elements (17) and (19), suppressed by one or more powers of $1/m_b$. The correction of order $1/m_b$ to (17) is equal to

$$\begin{aligned} & m_b^{n-1} \sum_{j=2}^n v_{\mu_2} \cdots \not{x}_{\mu_j} \cdots v_{\mu_n} \langle B(v) | \bar{h}_v^{(b)} \gamma_{\mu_1} (iD)_{\mu_j} h_v^{(b)} | B(v) \rangle \\ & + m_b^{n-2} \sum_{j=2}^n v_{\mu_2} \cdots \not{x}_{\mu_j} \cdots v_{\mu_n} \langle B(v) | \bar{h}_v^{(b)} \gamma_{\mu_1} \frac{i \vec{\not{D}}_{\perp}}{2} h_v^{(b)} | B(v) \rangle \\ & - m_b^{n-2} \sum_{j=2}^n v_{\mu_2} \cdots \not{x}_{\mu_j} \cdots v_{\mu_n} \langle B(v) | \bar{h}_v^{(b)} \frac{i \overleftarrow{\not{D}}_{\perp}}{2} \gamma_{\mu_1} h_v^{(b)} | B(v) \rangle \\ & + \frac{i}{2} m_b^{n-2} \sum_{j=2}^n \langle B(v) | \int d^4x T(\bar{h}_v^{(b)} (i \not{D}_{\perp})^2 h_v^{(b)})(x) (\bar{h}_v^{(b)} \gamma_{\mu_1} h_v^{(b)} | B(v) \rangle v_{\mu_2} \cdots \not{x}_{\mu_j} \cdots v_{\mu_j} . \end{aligned} \quad (25)$$

The matrix element in the first term can be written as

$$\langle B(v) | \bar{h}_v^{(b)} \gamma_{\mu_1} (iD)_{\mu_j} h_v^{(b)} | B(v) \rangle = A v_{\mu_1} v_{\mu_j} \quad (26)$$

with

$$A = \langle B(v) | \bar{h}_v^{(b)} (i v \cdot D) h_v^{(b)} | B(v) \rangle = 0 \quad (27)$$

where the equation of motion $i v \cdot D h_v^{(b)} = 0$ has been used. The other three terms have together the form of the $1/m_b$ correction to the matrix element of the vector current $\bar{b} \gamma_{\mu_1} b$, which is known to have no such corrections. Therefore the entire $1/m_b$ correction to the matrix element of the P operator vanishes. A similar argument shows that the $1/m_b$ corrections to the matrix elements of R vanish also. From this follows the conclusion that there are no $1/m_b$ corrections to the distribution functions of a heavy quark in a heavy hadron.

In principle the same line of reasoning will give also the $1/m_b^2$ corrections to the distribution functions. This will require the introduction in the operator product expansion (8) of some new twist three operators containing one gluon field tensor $F_{\mu\nu}$. Much simpler it is however to use the result for $T_{\mu\nu}$ given in the second paper of Ref. [1]. We will show that the answer obtained with the two methods for the

twist-2 contribution is the same. In the next section we discuss the corrections of order $1/m_b^2$ to the distribution functions of the heavy quarks.

3. Blok, Koyrakh, Shifman and Vainshtein have calculated $T_{\mu\nu}$ up to order $\bar{\Lambda}^2/m_b^2$ (Eqs.(A1,A2) in the second paper of Ref. [1]). Although they were interested in a timelike q , their expressions are also valid for a spacelike q , the case we need here. The results for $T_{1,2}$ are

$$\begin{aligned} \frac{1}{2m_B}T_1 &= \frac{1}{\Delta}[-v \cdot q + \frac{4m_b}{3}(K_b + G_b)] \\ &+ \frac{2m_b}{\Delta^2}\{-\frac{1}{3}G_b[-(v \cdot q)(4m_b - 3v \cdot q) + 2[(v \cdot q)^2 - q^2]] \\ &+ K_b[-(v \cdot q)^2 - \frac{2}{3}[(v \cdot q)^2 - q^2]]\} \\ &- \frac{8m_b^2}{3\Delta^3}K_b(v \cdot q)[(v \cdot q)^2 - q^2] + (q \rightarrow -q) \end{aligned} \quad (28)$$

and

$$\begin{aligned} \frac{1}{2m_B}T_2 &= \frac{2m_b}{\Delta}[1 + \frac{5}{3}(K_b + G_b)] + \frac{4m_b^2}{3\Delta^2}(7K_b + 5G_b)v \cdot q \\ &+ \frac{16m_b^3}{3\Delta^3}K_b[(v \cdot q)^2 - q^2] + (q \rightarrow -q) \end{aligned} \quad (29)$$

with

$$\Delta = -2m_b v \cdot q - Q^2 + i\epsilon. \quad (30)$$

The terms with $q \rightarrow -q$ have to be added to account for the necessary crossing property (3) of $T_{1,2}$. K_b and G_b are defined by

$$K_b = -\frac{1}{2m_B}\langle B(v)|\bar{h}_v^{(b)}\frac{(iD)^2}{2m_b^2}h_v^{(b)}|B(v)\rangle = -\frac{\lambda_1}{2m_b^2} = \frac{\mu_\pi^2}{2m_b^2} \quad (31)$$

$$G_b = \frac{1}{2m_B}\langle B(v)|\bar{h}_v^{(b)}\frac{g\sigma_{\alpha\beta}F^{\alpha\beta}}{4m_b^2}h_v^{(b)}|B(v)\rangle = -\frac{3\lambda_2}{2m_b^2} = -\frac{\mu_G^2}{2m_b^2} \quad (32)$$

and have the approximative values $G_b = -0.0077$ and $K_b = 0.01$. We have shown here the different names by which these quantities are denoted in the literature [1, 4, 5, 7, 11].

The expressions (28,29) are the first terms of a power series in the expansion parameters $\bar{\Lambda}/m_b$ and $\bar{\Lambda}^2/\Delta$. Because we are interested in evaluating these functions around the point $v \cdot q = 0$, the second expansion parameter is actually $\bar{\Lambda}^2/Q^2$.

We expand the expressions for $T_{1,2}$ into a power series in $(2m_B v \cdot q)/Q^2$ with the result for the coefficient of the n^{th} power

$$T_{1,n} = -2 \left(\frac{m_b}{m_B} \right)^{n-1} \left\{ 1 + (n-1) \frac{5G_b + (n+3)K_b}{3} + \frac{4}{3} \left(\frac{m_b^2}{Q^2} \right) [4(n+1)G_b + (n^2 + 3n + 4)K_b] \right\} \quad (33)$$

and

$$\frac{Q^2}{2m_B^2} T_{2,n} = -4 \left(\frac{m_b}{m_B} \right)^{n+1} \left\{ 1 + (n+1) \frac{5G_b + (n+5)K_b}{3} + \frac{4}{3} K_b (n+1)(n+2) \left(\frac{m_b^2}{Q^2} \right) \right\} \quad (34)$$

for n even (0,2,4,...) and $T_{i,n} = 0$ for n odd.

According to the discussion in the preceding Section, these coefficients are directly related to the $(n-1)^{th}$ moment of the respective structure functions. By performing an inverse Mellin transform or simply by trial, the functional dependence of $F_{1,2}(x)$, including these corrections, turns out to be of the form

$$\begin{aligned} F_1(x, Q^2) = & \left(1 + \frac{32}{3} (K_b + G_b) \frac{m_b^2}{Q^2} \right) \delta\left(x - \frac{m_b}{m_B}\right) \\ & - \frac{m_b}{m_B} \left(\frac{5}{3} (K_b + G_b) + 8(K_b + \frac{2}{3}G_b) \frac{m_b^2}{Q^2} \right) \delta'\left(x - \frac{m_b}{m_B}\right) \\ & + \frac{1}{3} K_b \left(\frac{m_b}{m_B} \right)^2 \left(1 + 4 \frac{m_b^2}{Q^2} \right) \delta''\left(x - \frac{m_b}{m_B}\right). \end{aligned} \quad (35)$$

and

$$\begin{aligned} \frac{1}{2} \left(\frac{m_B}{m_b} \right)^2 x F_2(x, Q^2) = & \left(1 + 4K_b + \frac{10}{3}G_b + 8K_b \frac{m_b^2}{Q^2} \right) \delta\left(x - \frac{m_b}{m_B}\right) \\ & - \frac{m_b}{m_B} \left(3K_b + \frac{5}{3}G_b + 8K_b \frac{m_b^2}{Q^2} \right) \delta'\left(x - \frac{m_b}{m_B}\right) \\ & + \frac{1}{3} K_b \left(\frac{m_b}{m_B} \right)^2 \left(1 + 4 \frac{m_b^2}{Q^2} \right) \delta''\left(x - \frac{m_b}{m_B}\right). \end{aligned} \quad (36)$$

This is the same as what one obtains by directly taking the discontinuity of (28,29) by using the well-known identity

$$\text{Im} \frac{1}{x + i\epsilon} = -\pi \delta(x). \quad (37)$$

The Eqs. (35,36) are all what QCD predicts about the structure functions of a heavy quark in a B meson at order $\bar{\Lambda}^2/m_b^2, \bar{\Lambda}^2/Q^2$. They have a rather singular dependence of the Bjorken variable x , given by a δ function and its derivatives. These singularities are similar to what has been recently obtained for the shape of the electron spectrum near its end-point in $B \rightarrow X_u e \bar{\nu}$ or for the photon spectrum in $B \rightarrow X_s \gamma$ decays. An interesting feature of these predictions is that, although the support of the delta functions is restrained to a single point $x = m_b/m_B$, they do simulate a spreading of the structure functions from the parton-model prediction. For example, one can compute the dispersion of the x variable σ_x for the structure function $F_1(x)$:

$$\sigma_x^2 = \langle x^2 \rangle_{F_1} - \langle x \rangle_{F_1}^2 = \frac{m_b^2}{m_B^2} \left(\frac{2}{3} K_b - 8 \left(K_b + \frac{4}{3} G_b \right) \frac{m_b^2}{Q^2} \right) \quad (38)$$

which is nonvanishing and is of the order $\bar{\Lambda}/m_b$.

Of course, the physical structure function does not look like a delta function and derivatives thereof, but has a smooth shape which extends beyond the parton model boundary¹. We will show that the sum of the most singular terms in the expansions (35,36) to all orders in $\bar{\Lambda}/m_b$ generates a “shape function” which can be considered as an approximation to the physical structure function. The argument goes in close analogy to the analysis of Neubert of the photon spectrum in $B \rightarrow X_s \gamma$ decays. However, the details of the derivation are somewhat different, as appropriate to the problem at hand.

We will consider in the following the scaling limit $Q^2 \rightarrow \infty$ when the distribution functions depend only on the Bjorken variable x . Note that these are not the distribution functions which would be measured in a hypothetical electroproduction experiment on B mesons, since the relevant matrix elements are evaluated at the heavy quark scale and the observable distribution functions would have to be obtained by evolution to the large momentum scale Q^2 . However, we will have nothing to say about this evolution and therefore all our results refer exclusively to the “boundary data” for the distribution functions at the scale $\mu^2 = m_b^2$.

In this limit there is only one independent structure function, since the two functions $F_1(x)$ and $F_2(x)$ are connected by the Callan-Gross relation $F_2 = 2xF_1$. One can easily convince himself that the predictions (35,36) satisfy this equality in the limit $Q^2 \rightarrow \infty$. It is only violated at first order in α_s . Therefore, we will proceed further with $F_1(x)$ and call it simply “the distribution function”.

The physical distribution function in the scaling limit $F_1(x)$ can be represented, when convoluted with an arbitrary smooth function, as the sum of a delta function

¹We assume that the rapid variations in the physical structure functions due to the resonances have been smoothed out.

and its derivatives:

$$F_1(x) = \sum_{i=0}^{\infty} (-1)^i \frac{M_i}{i!} \delta^{(i)}\left(x - \frac{m_b}{m_B}\right) \quad (39)$$

where the coefficients M_i are the moments of $F_1(x)$

$$M_i = \int_0^1 dx \left(x - \frac{m_b}{m_B}\right)^i F_1(x). \quad (40)$$

The first few moments M_i can be extracted from Eq.(35)

$$M_0 = 1 \quad (41)$$

$$M_1 = \frac{5}{3} \frac{m_b}{m_B} (K_b + G_b) \quad (42)$$

$$M_2 = \frac{2}{3} \frac{m_b^2}{m_B^2} K_b \quad (43)$$

Each of the moments M_i can be expanded in a power series in $\bar{\Lambda}/m_b$ and it is reasonable to assume that keeping only the leading term in this expansion provides a good first approximation to the description of $F_1(x)$. This is equivalent to keeping only the leading singularities in (38) (the smallest power of $\bar{\Lambda}/m_b$ corresponding to a derivative of a given order of the δ function is the same as the highest order of the derivative corresponding to a given order in $\bar{\Lambda}/m_b$).

In the scaling limit $Q^2 \rightarrow \infty$ only the operator P in (9) gives a nonvanishing contribution to the distribution functions, since it is the only one which contains twist-2 operators. The operator R and any other operators involving the gluon field tensor give contributions suppressed by one or more powers of Q^2 . The matrix elements of P have the form

$$\langle B(v) | P_{\mu_1 \mu_2 \dots \mu_n} | B(v) \rangle = C v_{\mu_1} v_{\mu_2} \dots v_{\mu_n} + \text{terms containing } g_{\mu\nu} \text{'s} \quad (44)$$

We are only interested in the part proportional to v 's because the terms with $g_{\mu\nu}$ vanish in the scaling limit. Let us try to extract as much information as possible about the constant C to all orders in $1/m_b$. It has an expansion of the form

$$C = 2m_B m_b^{n-1} \left(1 + \epsilon C_1 + \epsilon^2 C_2 + \dots\right) \quad (45)$$

with

$$\epsilon = \frac{\bar{\Lambda}}{m_b} \quad (46)$$

A direct calculation by making use of the methods exposed in Section 2 gives

$$\epsilon C_1 = 0 \quad (47)$$

$$\epsilon^2 C_2 = \frac{2}{3} K_b \frac{(n-1)(n-2)}{2} + \frac{2}{3} (K_b + G_b)(n-1) + (K_b + G_b)(n-1) \quad (48)$$

The first relation expresses the vanishing of the $1/m_b$ corrections to the structure functions, noted in the preceding Section. The three terms in the second relation have respectively, the following origin: i) the first appears from the matrix element $\langle B(v)|(iD)_\mu(iD)_\nu|B(v)\rangle$ resulting from the expansion of (17); ii) the second appears from the correction to the B field (16) at order $1/m_b$ and iii) the third appears from the insertion of the $1/m_b$ term in the HQET Lagrangian.

If the expression (44) together with (45,47,48) are introduced into Eq.(8) and the result for T_1 expanded in a power series in $(2m_B v \cdot q)/Q^2$, the coefficient of the n^{th} power which is obtained coincides with Eq.(33) (in the limit $Q^2 \rightarrow \infty$). This establishes the equivalence of the two approaches to this order in $\bar{\Lambda}/m_b$.

The constant C is directly related to the $(n-1)^{th}$ moment of the structure function $F_1(x)$, as can be seen upon inserting (45) into the general formula (8):

$$\int_0^1 dx x^{n-1} F_1(x) = \left(\frac{m_b}{m_B}\right)^{n-1} (1 + \epsilon C_1 + \epsilon^2 C_2 + \dots) \quad (49)$$

For a given order in ϵ (say, ϵ^k), the most singular terms in F_1 correspond to those terms in the coefficient C_k which have the highest power of n . For example, the term with the highest power of n in C_2 is $\frac{1}{3}K_b n^2$ and it corresponds to the contribution $\frac{1}{3}K_b \left(\frac{m_b}{m_B}\right)^2 \delta''(x - \frac{m_b}{m_B})$ in $F_1(x)$, Eq.(35). As said before, this term originated from simply expanding the brackets in

$$\langle B(v)|\bar{B}(m_b v + iD)_{\mu_1}(m_b v + iD)_{\mu_2} \cdots (m_b v + iD)_{\mu_n} B|B(v)\rangle, \quad (50)$$

replacing B by the HQET field $h_v^{(b)}$ and forgetting about lagrangian insertions. We will show now by a simple counting argument that this holds true generally and that the leading terms in n are correctly reproduced by retaining only these contributions, to all orders in ϵ .

The contribution to C_k of these terms can be written as

$$\frac{1}{2m_B m^k} \sum \langle B(v)|\bar{h}_v^{(b)} \gamma_{\mu_1} (iD)_{\alpha_1} (iD)_{\alpha_2} \cdots (iD)_{\alpha_k} h_v^{(b)} |B(v)\rangle v_{\alpha_{k+1}} \cdots v_{\alpha_n} \quad (51)$$

where the sum runs over all possible ways to choose the k derivatives out of a total of $(n-1)$ possibilities, and contains thus $\frac{(n-1)!}{k!(n-k-1)!}$ terms.

Let us consider now a contribution to C_k arising from corrections to the B field. We write the expansion (16) of B in powers of $1/m_b$ in a compact form as

$$B(x) = \sum_{n=0}^{\infty} \frac{B_n}{m_b^n} h_v^{(b)}(x) \quad (52)$$

with $B_0 = 1$, $B_1 = (i\not{D})/2$, etc. Then the desired contribution reads

$$\frac{1}{2m_B m^k} \sum_{0 \leq i+j \leq k-1} \langle B(v)|\bar{h}_v^{(b)} \bar{B}_i \gamma_{\mu_1} (iD)_{\alpha_1} (iD)_{\alpha_2} \cdots (iD)_{\alpha_{k-i-j}} B_j h_v^{(b)} |B(v)\rangle v_{\alpha_{k-i-j+1}} \cdots v_{\alpha_n} \quad (53)$$

and the sum contains

$$\sum_{i=0}^{k-1} \sum_{j=0}^{k-i-1} \frac{(n-1)!}{(k-i-j)!(n-k+i+j-1)!} = (n-1)! \sum_{r=0}^{k-1} \frac{r+1}{(k-r)!(n-k+r-1)!} \quad (54)$$

terms. We can omit the terms with $i+j=k$ because this is a correction to the matrix element of the vector current which is known to vanish. We have included here also the preceding case as the term with $i=j=0$ respectively $r=i+j=0$. Assuming all the matrix elements in (53) to have about the same order of magnitude, one can see that the total contribution from the matrix elements without corrections to the B field ($r=0$) scales like n^k for a given order k in the ϵ expansion. The next contributions, containing a suppression by $1/m_b$ from a correction to the heavy quark field ($r=1$), scale as n^{k-1} , and so on. The matrix elements containing insertions of higher-dimensional terms in the HQET lagrangian can be shown in a similar fashion to give only subleading contributions in n . This proves the above-mentioned assertion that the most singular terms in $F_1(x)$ are correctly reproduced by replacing

$$\begin{aligned} & \langle B(v) | P_{\mu_1 \mu_2 \dots \mu_n} | B(v) \rangle \rightarrow \\ & v_{\mu_1} \langle B(v) | \bar{h}_v^{(b)} (m_b v + iD)_{\mu_2} (m_b v + iD)_{\mu_3} \dots (m_b v + iD)_{\mu_n} h_v^{(b)} | B(v) \rangle \\ & + \text{symmetrization w.r.t. all indices} \end{aligned} \quad (55)$$

We define the dimensionless constants $a_n \simeq 1$ through

$$\frac{1}{2m_B} \langle B(v) | \bar{h}_v^{(b)} (iD)_{\mu_1} (iD)_{\mu_2} \dots (iD)_{\mu_n} h_v^{(b)} | B(v) \rangle = a_n \bar{\Lambda}^n v_{\mu_1} v_{\mu_2} \dots v_{\mu_n} + \dots \quad (56)$$

The first few a_i are

$$a_0 = 1 \quad (57)$$

$$\epsilon a_1 = 0 \quad (58)$$

$$\epsilon^2 a_2 = \frac{2}{3} K_b. \quad (59)$$

In terms of these constants the matrix element (55) can be written as

$$\begin{aligned} & v_{\mu_1} \langle B(v) | \bar{h}_v^{(b)} (m_b v + iD)_{\mu_2} (m_b v + iD)_{\mu_3} \dots (m_b v + iD)_{\mu_n} h_v^{(b)} | B(v) \rangle \\ & + \text{symmetrization w.r.t. all indices} = 2m_B m_b^{n-1} \sum_{i=0}^{n-1} \frac{(n-1)!}{i!(n-i-1)!} a_i \frac{\bar{\Lambda}^i}{m_b^i} v_{\mu_1} v_{\mu_2} \dots v_{\mu_n}. \end{aligned} \quad (60)$$

Inserting this in Eq.(8) gives for the moments of $F_1(x)$ (only the part containing the leading singularities, the so-called “shape function”)

$$\int_0^1 dx x^{n-1} F_1(x) = \left(\frac{m_b}{m_B} \right)^{n-1} \sum_{i=0}^{n-1} \frac{(n-1)!}{i!(n-i-1)!} a_i \epsilon^i. \quad (61)$$

The moments M_i with respect to the parton-model point $x = m_b/m_B$ which appear in (39) can be obtained from the above ones by a simple expansion

$$\begin{aligned} M_{n-1} &= \int_0^1 dx \left(x - \frac{m_b}{m_B}\right)^{n-1} F_1(x) = \sum_{i=0}^{n-1} \frac{(n-1)!}{i!(n-i-1)!} \left(-\frac{m_b}{m_B}\right)^{n-i-1} \int_0^1 dx x^i F_1(x) \\ &= \left(\frac{m_b}{m_B}\right)^{n-1} \sum_{i=0}^{n-1} \sum_{j=0}^i (-1)^{n-i-1} \frac{(n-1)!}{j!(n-i-1)!(i-j)!} a_j \epsilon^j = \left(\frac{m_b}{m_B}\right)^{n-1} \epsilon^{n-1} a_{n-1}, \end{aligned} \quad (62)$$

where the final result has been obtained by interchanging the summation order. It has been shown [4, 5, 7] that the line shape in the $B \rightarrow X_s \gamma$ decays and the end-point electron spectrum in the semileptonic $B \rightarrow X_u$ decays are essentially determined by one common universal function whose moments are related to the nonperturbative matrix elements a_i in a fashion similar to (62). Therefore the result (62) is remarkable, as it shows that the same universal function gives also the heavy quark structure functions for electroproduction on a B meson.

Most probably, these structure functions will never be measured and therefore our results are of little practical value. Nevertheless, we believe that they are interesting in themselves because they show how completely different phenomena involving heavy quarks are governed by similar distribution functions. Finally, we mention the related problem, of considerable experimental interest, of the fragmentation functions for heavy quarks into heavy hadrons, for which one cannot make use of the powerful formalism of the operator product expansion. One can hope that similar methods could be used to study the shape of these functions, and set constraints on the possible structure of their nonperturbative contributions.

4. In this Section we derive two new sum rules for the nonperturbative functions which describe the form-factors for heavy-to-heavy transitions. These sum rules are deduced for the process $\gamma^* + B \rightarrow X_b$, with X_b an arbitrary excited state of the B meson, but they can be extended by using heavy quark symmetry for the case of greater interest $B \rightarrow X_c + W$, with X_c an excited state of the D meson.

We will relax the assumption $Q^2 \rightarrow \infty$ and keep Q^2 arbitrary (but still much larger than Λ_{QCD}). Let us examine the form of the physical structure functions $F_{1,2}(x, Q^2)$. In the neighbourhood of $x = 1$ (near the end of the cuts) they have a rapid variation, with spikes corresponding to each resonance X_b superposed over a smooth background due to two- and many-body intermediate states, such as $B\pi$. The contribution of a particular resonance of mass m_X to $W_{\mu\nu}$ can be obtained from (4) to be given by

$$W_{\mu\nu}^{(X)}(p \cdot q, Q^2) = \frac{1}{4} \delta[p \cdot q - \frac{1}{2}(m_B^2 - m_X^2 - Q^2)] \times \quad (63)$$

$$\langle B(v)|(\bar{b}\gamma_\mu b)(0)|X_b(v')\rangle\langle X_b(v')|(\bar{b}\gamma_\nu b)(0)|B(v)\rangle$$

where the velocity v' is given by

$$v \cdot v' = \frac{m_B^2 + m_X^2 + Q^2}{2m_B m_X}. \quad (64)$$

The matrix elements can be expanded in powers of $1/m_b$ by using heavy quark effective theory methods and the resulting structure functions $F_{1,2}$ are expressed in terms of the usual Bjorken variable x by using

$$\delta[p \cdot q - \frac{1}{2}(m_B^2 - m_X^2 - Q^2)] = \frac{2x_X^2}{Q^2} \delta(x - x_X) \quad (65)$$

with

$$x_X = \frac{Q^2}{Q^2 - m_B^2 + m_X^2}. \quad (66)$$

The key ingredient which makes possible writing a sum rule is the positivity property of $F_{1,2}$ [12], which expresses the fact that each individual intermediate state in (4) contributes a positive quantity. Equating the OPE result for the moments (33,34) with the moments of the structure functions written as sums over intermediate states gives an upper bound on any partial sum of the latter.

On the tip of the cut, $m_X = m_B$ and $v \cdot v' = 1 + Q^2/(2m_B^2)$. The velocity transfer $v \cdot v'$ decreases as we move along the cut until it reaches a minimum $(v \cdot v')_{min} = \sqrt{1 + Q^2/m_B^2}$ for $m_X = m_B \sqrt{1 + Q^2/m_B^2}$ after which it starts increasing. Therefore the various intermediate states will appear in the sum rule with a different velocity transfer $v \cdot v'$.

From (63) we get

$$W_{\mu\nu}^X(x, Q^2) = \frac{Q^2}{2(Q^2 + m_X^2 - m_B^2)^2} \delta(x - x_X) \langle B(v)|(\bar{b}\gamma_\mu b)(0)|X_b(v')\rangle\langle X_b(v')|(\bar{b}\gamma_\nu b)(0)|B(v)\rangle \quad (67)$$

or, in explicit form for the s- and p-wave mesons

$$F_1^i(x, Q^2) = \frac{Q^2}{2(Q^2 + m_{X_i}^2 - m_B^2)^2} \delta(x - x_{X_i}) R^i \quad (68)$$

with

- B meson $R = 0$.
- B* meson $R = m_B m_{B^*} (\omega^2 - 1) |\xi + \frac{1}{m_b} (2\chi_1 - 2(\omega - 1)\chi_2 + 4\chi_3 + \bar{\Lambda}\xi - \xi_3)|^2$.

- B₀ meson ($s_\ell^{\pi_\ell} = 1/2^+, s^\pi = 0^+$) $R = 0$.
- B₁ meson ($s_\ell^{\pi_\ell} = 1/2^+, s^\pi = 1^+$) $R = \frac{1}{3}m_B m_{B_1}(\omega - 1)^2 |\xi_{1/2}|^2$.
- B'₁ meson ($s_\ell^{\pi_\ell} = 3/2^+, s^\pi = 1^+$) $R = \frac{1}{6}m_B m_{B'_1}(\omega^2 - 1)^2 |\xi_{3/2}|^2$.
- B₂ meson ($s_\ell^{\pi_\ell} = 3/2^+, s^\pi = 2^+$) $R = \frac{1}{2}m_B m_{B_2}(\omega^2 - 1)^2 |\xi_{3/2}|^2$.

and for $xF_2(x, Q^2)$

$$xF_2^i(x, Q^2) = \frac{Q^4}{(Q^2 + m_{X_i}^2 - m_B^2)^2} \delta(x - x_{X_i}) S^i \quad (69)$$

with

- B meson $S = |\xi + \frac{2}{m_b}(\chi_1 - 2(\omega - 1)\chi_2 + 6\chi_3)|^2$.
- B* meson $S = \frac{m_{B^*}}{4m_B} \left(-1 - \frac{m_B^2}{m_{B^*}^2} + 2\frac{m_B}{m_{B^*}}\omega \right) |\xi + \frac{1}{m_b}(2\chi_1 - 2(\omega - 1)\chi_2 + 4\chi_3 + \bar{\Lambda}\xi - \xi_3)|^2$.
- B₀ meson ($s_\ell^{\pi_\ell} = 1/2^+, s^\pi = 0^+$) $S = 0$.
- B₁ meson ($s_\ell^{\pi_\ell} = 1/2^+, s^\pi = 1^+$) $S = \frac{1}{6}(\omega - 1)|\xi_{1/2}|^2$.
- B'₁ meson ($s_\ell^{\pi_\ell} = 3/2^+, s^\pi = 1^+$) $S = \frac{m_{B'_1}}{8m_B} \left(1 + \frac{m_B^2}{m_{B'_1}^2} + \frac{2m_B}{3m_{B'_1}}(\omega + 4) \right) (\omega^2 - 1)|\xi_{3/2}|^2$.
- B₂ meson ($s_\ell^{\pi_\ell} = 3/2^+, s^\pi = 2^+$) $S = \frac{m_{B_2}}{8m_B} \left(-1 - \frac{m_B^2}{m_{B_2}^2} + 2\frac{m_B}{m_{B_2}}\omega \right) (\omega^2 - 1)|\xi_{3/2}|^2$.

Here ω stands for the velocity change $v \cdot v'$. $\xi(\omega), \xi_{1/2, 3/2}(\omega)$ are the Isgur-Wise functions which describe the form-factors of the transitions from the respective multiplets to the lowest-lying one². For the s-wave multiplet we have included also the contribution from the subleading form-factors $\chi_{1,2,3}, \xi_3$, which are defined as in [14, 11]. For example, if we assume for the moment that the multiplets are degenerate, their contribution to $F_{1,2}$ takes the form

$$\begin{aligned} F_1^{res}(x, Q^2) &= \frac{1 + \omega}{2} \left| \xi + \frac{1}{m_b}(2\chi_1 - 2(\omega - 1)\chi_2 + 4\chi_3 + \bar{\Lambda}\xi - \xi_3) \right|^2 \delta(x - 1) \\ &+ \frac{Q^2[Q^2 + (m_{B_1} - m_B)^2]}{6[Q^2 + (m_{B_1}^2 - m_B^2)]^2} (\omega - 1) |\xi_{1/2}|^2 \delta(x - x_{B_1}) \\ &+ \frac{Q^2[Q^2 + (m_{B_2} - m_B)^2]}{3[Q^2 + (m_{B_2}^2 - m_B^2)]^2} (\omega - 1)(1 + \omega)^2 |\xi_{3/2}|^2 \delta(x - x_{B_2}) + \dots \end{aligned} \quad (70)$$

² $\xi_{1/2, 3/2}$ are related to the form-factors used in [13] by $\xi_{1/2} = 2\sqrt{3}\tau_{1/2}$ and $\xi_{3/2} = \sqrt{3}\tau_{3/2}$.

and

$$\begin{aligned}
xF_2^{res}(x, Q^2) &= \frac{1+\omega}{2} |\xi|^2 \delta(x-1) \\
&+ \frac{Q^4}{6[Q^2 + (m_{B_1}^2 - m_B^2)]^2} (\omega-1) |\xi_{1/2}|^2 \delta(x-x_{B_1}) \\
&+ \frac{Q^4}{3[Q^2 + (m_{B_2}^2 - m_B^2)]^2} (\omega-1)(1+\omega)^2 |\xi_{3/2}|^2 \delta(x-x_{B_2}) + \dots
\end{aligned} \tag{71}$$

However, since we work to an arbitrary order in $1/m_b$, we will consider henceforth the masses of the two members of each multiplet as different.

The sum rules result when the moments of $F_{1,2}$ expressed as a sum over resonances are equated with the QCD prediction

$$\begin{aligned}
\int_0^1 dx x^{n-1} F_1(x, Q^2) &= \left(\frac{m_b}{m_B}\right)^{n-1} \left\{ 1 + (n-1) \frac{5G_b + (n+3)K_b}{3} \right. \\
&\quad \left. + \frac{4}{3} \left(\frac{m_b^2}{Q^2}\right) [4(n+1)G_b + (n^2 + 3n + 4)K_b] + \dots \right\}
\end{aligned} \tag{72}$$

and

$$\begin{aligned}
\int_0^1 dx x^n F_2(x, Q^2) &= \left(\frac{m_b}{m_B}\right)^{n+1} \left\{ 1 + (n+1) \frac{5G_b + (n+5)K_b}{3} \right. \\
&\quad \left. + \frac{4}{3} K_b (n+1)(n+2) \left(\frac{m_b^2}{Q^2}\right) + \dots \right\}
\end{aligned} \tag{73}$$

At this point a serious problem emerges which might diminish the usefulness of the sum rules. The most interesting region where one would like to apply these sum rules is in the vicinity of the equal-velocity point $v \cdot v' = 1$, which corresponds to $Q^2 = 0$. But, as one can see from (72,73), the $1/m_b^2$ corrections become very large in this limit, and therefore one can trust no longer the expansion in powers of $1/m_b$. This happens because the expansion parameter in this problem is $\bar{\Lambda}/Q$. For example, if we require as a limit of applicability that $\bar{\Lambda}/Q = 0.5$, this would correspond to $v \cdot v' \simeq 1.018$ on the tip of the cut.

There exists yet another way to apply the sum rules, which allows one to consider practically any value of $v \cdot v'$. This can be done if the b quark mass is considered as a free parameter. Then one can proceed in the following way: i) the ratio Q^2/m_B^2 must be kept fixed to a value given by the velocity change $v \cdot v'$ one wishes to investigate, according to (64); ii) m_B and Q must be sent simultaneously to infinity (compared to $\bar{\Lambda}$, which is of course fixed); iii) the result is expanded in inverse powers of m_B . This procedure makes sure that the higher order corrections in (72,73) stay smaller than the leading term.

In the limiting case when $Q^2, m_B \rightarrow \infty$ with $Q^2/m_B^2 = 2(\omega - 1)$ the two sum rules for F_1 and xF_2 reduce to the usual Bjorken sum rule [13]:

$$1 = \frac{1+\omega}{2}|\xi|^2 + \frac{1}{6}(\omega-1)|\xi_{1/2}|^2 + \frac{1}{3}(\omega-1)(1+\omega)^2|\xi_{3/2}|^2 + \dots \quad (74)$$

For each n , a different sum rule is obtained. Higher values of n suppress the contributions of the excited states, since x_X is smaller for these. We note that one cannot write one sum rule for each power of $1/m_b$, since the various contributions will not contribute in general with the same sign, so we cannot set constraints on the subleading form-factors appearing to a given order in the heavy mass expansion. Rather, the sum rules can be used to test the consistency of a set of calculated form-factors: the total contribution of a number of resonant states, calculated to a given order in $1/m_b$, must not exceed the QCD prediction for the sum of all excited states, calculated to the same order in $1/m_b$.

Note added. After completing this work, the paper [15] appeared, which extends the results in [4, 5, 6, 7] to cases when the final quark mass is nonvanishing. Also, a different derivation of the Bjorken sum rule has been recently given in [16].

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Figure Caption

Fig.1 The analytical structure of the forward Compton scattering amplitude $T_{\mu\nu}$ as a function of $p \cdot q$ for given Q^2 .

References

- [1] I.I.Bigi, M.Shifman, N.G.Uraltsev and A.I.Vainshtein, *Phys.Rev.Lett.***71** (1993) 496
B.Blok, L.Koyrakh, M.Shifman and A.I.Vainshtein, NSF-ITP-93-68, TPI-MINN-93/33-T, UMN-TH-1208/93, July 1993
A.Manohar and M.Wise, UCSD/PTH 93-14, CALT-68-1883, August 1993
T.Mannel, IKDA 93/26, September 1993
- [2] A.F.Falk, M.Luke and M.J.Savage, SLAC preprint SLAC-PUB-6317 (1993)

- [3] J.Chay, H.Georgi and B.Grinstein, *Phys.Lett.***B247** (1990) 399
- [4] M.Neubert, CERN-TH.7087/93, November 1993
- [5] M.Neubert, CERN-TH.7113/93, December 1993
- [6] A.F.Falk, E.Jenkins, A.V.Manohar and M.B.Wise, UCSD-PTH-93-38, December 1993
- [7] I.I.Bigi, M.A.Shifman, N.G.Uraltsev and A.I.Vainshtein, TPI-MINN-93/60-T, UMN-TH-1231-93, CERN-TH.7129/93, December 1993
- [8] J.D.Bjorken, SLAC preprint SLAC-PUB-5278 (1990)
J.D.Bjorken, I.Dunietz and J.Taron, *Nucl.Phys.***B 371**(1992)111
- [9] R.L.Jaffe and L.Randall, *Nucl.Phys.***B412**(1994)79
- [10] N.Christ, B.Hasslacher and A.H.Mueller, *Phys.Rev.* **D6**(1972)3543
- [11] A.F.Falk and M.Neubert, *Phys.Rev.***D47**(1993)2965
- [12] C.Itzykson and J.Zuber, *Quantum Field Theory*, Mc Graw-Hill, N.Y. 1980
- [13] N.Isgur and M.B.Wise, *Phys.Rev.***D43**(1991)819
N.Isgur, M.B.Wise and M.Youssefmir, *Phys.Lett.***B254**(1991)215
W.Roberts, *Nucl.Phys.***B389**(1993)549
- [14] M.E.Luke, *Phys.Lett.***B252**(1990)447
- [15] T.Mannel and M.Neubert, CERN-TH.7156/94, February 1994
- [16] T.D.Cohen and J.Milana, UMPP #94-086, February 1994

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